

# Contractible groups and linear dilatation structures

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## Abstract

A dilatation structure on a metric space, is a notion in between a group and a differential structure. The basic objects of a dilatation structure are dilatations (or contractions). The axioms of a dilatation structure set the rules of interaction between different dilatations.

There are two notions of linearity associated to dilatation structures: the linearity of a function between two dilatation structures and the linearity of the dilatation structure itself.

Our main result here is a characterization of contractible groups in terms of dilatation structures. To a normed conical group (normed contractible group) we can naturally associate a linear dilatation structure. Conversely, any linear and strong dilatation structure comes from the dilatation structure of a normed contractible group.

**Keywords:** contractible groups, Carnot groups, dilatation structures, metric tangent spaces

**MSC classes:** 22E20; 20E36; 20F65; 22A10; 51F99

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## 1 Introduction

Dilatation structures on metric spaces, introduced in [3], describe the approximate self-similarity properties of a metric space. A dilatation structure is a notion related, but more general, to groups and differential structures.

The basic objects of a dilatation structure are dilatations (or contractions). The axioms of a dilatation structure set the rules of interaction between different dilatations.

A metric space  $(X, d)$  which admits a strong dilatation structure (definition 3.2) has a metric tangent space at any point  $x \in X$  (theorem 4.2), and any such metric tangent space has an algebraic structure of a conical group (theorem 4.3). Conical groups are particular examples of contraction groups. The structure of contraction groups is known in some detail, due to Siebert [9], Wang [10], Glöckner and Willis [6], Glöckner [5] and references therein.

By a classical result of Siebert [9] proposition 5.4, we can characterize the algebraic structure of the metric tangent spaces associated to dilatation structures of a certain kind: they are Carnot groups, that is simply connected Lie groups whose Lie algebra admits a positive graduation (corollary 4.7).

Carnot groups appear in many situations, in particular in relation with sub-riemannian geometry cf. Bellaïche [1], groups with polynomial growth cf. Gromov [7], or Margulis type rigidity results cf. Pansu [8]. It is part of the author program of research to show that dilatation structures are natural objects in all these mathematical subjects. In this respect the corollary 4.7 represents a generalization of difficult results in sub-riemannian geometry concerning the structure of the metric tangent space at a point of a regular subriemannian manifold.

Linearity is also a property which can be explained with the help of a dilatation structure. In the second section of the paper we explain why linearity can be casted in terms of dilatations. There are in fact two kinds of linearity: the linearity of a function between two dilatation structures (definition 5.1) and the linearity of the dilatation structure itself (definition 5.7).

Our main result here is a characterization of contraction groups in terms of dilatation structures. To a normed conical group (normed contraction group) we can naturally associate a linear dilatation structure (proposition 5.8). Conversely, by theorem 5.11 any linear and strong dilatation structure comes from the dilatation structure of a normed contraction group.

## 2 Linear structure in terms of dilatations

Linearity is a basic property related to vector spaces. For example, if  $\mathbb{V}$  is a real, finite dimensional vector space then a transformation  $A : \mathbb{V} \rightarrow \mathbb{V}$  is linear if it is a morphism of groups  $A : (\mathbb{V}, +) \rightarrow (\mathbb{V}, +)$  and homogeneous with respect to positive scalars. Furthermore, in a normed vector space we can speak about linear continuous transformations.

A transformation is affine if it is a composition of a translation with a linear transformation. In this paper we shall use the umbrella name "linear" for affine transformations too.

We try here to explain that linearity property can be entirely phrased in terms of dilatations of the vector space  $\mathbb{V}$ .

For the vector space  $\mathbb{V}$ , the dilatation based at  $x \in \mathbb{V}$ , of coefficient  $\varepsilon > 0$ , is the function

$$\delta_\varepsilon^x : \mathbb{V} \rightarrow \mathbb{V} \quad , \quad \delta_\varepsilon^x y = x + \varepsilon(-x + y) \quad .$$

For fixed  $x$  the dilatations based at  $x$  form a one parameter group which contracts any bounded neighbourhood of  $x$  to a point, uniformly with respect to  $x$ .

The algebraic structure of  $\mathbb{V}$  is encoded in dilatations. Indeed, using dilatations we can recover the operation of addition and multiplication by scalars.

For  $x, u, v \in \mathbb{V}$  and  $\varepsilon > 0$  we define the following compositions of dilatations:

$$\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v \quad , \quad \Sigma_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^{\delta_\varepsilon^x u}(v) \quad , \quad \text{inv}_\varepsilon^x(u) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} x \quad . \quad (2.0.1)$$

The meaning of this functions becomes clear if we compute:

$$\begin{aligned} \Delta_\varepsilon^x(u, v) &= x + \varepsilon(-x + u) + (-u + v) \quad , \\ \Sigma_\varepsilon^x(u, v) &= u + \varepsilon(-u + x) + (-x + v) \quad , \\ \text{inv}_\varepsilon^x(u) &= x + \varepsilon(-x + u) + (-u + x) \quad . \end{aligned}$$

As  $\varepsilon \rightarrow 0$  we have the limits:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon^x(u, v) &= \Delta^x(u, v) = x + (-u + v) \quad , \\ \lim_{\varepsilon \rightarrow 0} \Sigma_\varepsilon^x(u, v) &= \Sigma^x(u, v) = u + (-x + v) \quad , \\ \lim_{\varepsilon \rightarrow 0} \text{inv}_\varepsilon^x(u) &= \text{inv}^x(u) = x - u + x \quad , \end{aligned}$$

uniform with respect to  $x, u, v$  in bounded sets. The function  $\Sigma^x(\cdot, \cdot)$  is a group operation, namely the addition operation translated such that the neutral element is  $x$ . Thus, for  $x = 0$ , we recover the usual addition operation. The function  $\text{inv}^x(\cdot)$  is the inverse function with respect to addition, and  $\Delta^x(\cdot, \cdot)$  is the difference function.

Notice that for fixed  $x, \varepsilon$  the function  $\Sigma_\varepsilon^x(\cdot, \cdot)$  is not a group operation, first of all because it is not associative. Nevertheless, this function satisfies a shifted associativity property, namely (see theorem 4.1)

$$\Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u, v), w) = \Sigma_\varepsilon^x(u, \Sigma_\varepsilon^{\delta_\varepsilon^x u}(v, w)) \quad .$$

Also, the inverse function  $\text{inv}_\varepsilon^x$  is not involutive, but shifted involutive (theorem 4.1):

$$\text{inv}_\varepsilon^{\delta_\varepsilon^x u}(\text{inv}_\varepsilon^x u) = u \quad .$$

Affine continuous transformations  $A : \mathbb{V} \rightarrow \mathbb{V}$  admit the following description in terms of dilatations. (We could dispense of continuity hypothesis in this situation, but we want to illustrate a general point of view, described further in the paper).

**Proposition 2.1** *A continuous transformation  $A : \mathbb{V} \rightarrow \mathbb{V}$  is affine if and only if for any  $\varepsilon \in (0, 1)$ ,  $x, y \in \mathbb{V}$  we have*

$$A\delta_\varepsilon^x y = \delta_\varepsilon^{Ax} Ay \quad . \quad (2.0.2)$$

The proof is a straightforward consequence of representation formulæ (2.0.1) for the addition, difference and inverse operations in terms of dilatations.

### 3 Dilatation structures

We present here a brief introduction into the subject of dilatation structures. For more details see Buliga [3]. The results with proofs are new.

#### 3.1 Notations

Let  $\Gamma$  be a topological separated commutative group endowed with a continuous group morphism

$$\nu : \Gamma \rightarrow (0, +\infty)$$

with  $\inf \nu(\Gamma) = 0$ . Here  $(0, +\infty)$  is taken as a group with multiplication. The neutral element of  $\Gamma$  is denoted by 1. We use the multiplicative notation for the operation in  $\Gamma$ .

The morphism  $\nu$  defines an invariant topological filter on  $\Gamma$  (equivalently, an end). Indeed, this is the filter generated by the open sets  $\nu^{-1}(0, a)$ ,  $a > 0$ . From now on we shall name this topological filter (end) by "0" and we shall write  $\varepsilon \in \Gamma \rightarrow 0$  for  $\nu(\varepsilon) \in (0, +\infty) \rightarrow 0$ .

The set  $\Gamma_1 = \nu^{-1}(0, 1]$  is a semigroup. We note  $\bar{\Gamma}_1 = \Gamma_1 \cup \{0\}$ . On the set  $\bar{\Gamma} = \Gamma \cup \{0\}$  we extend the operation on  $\Gamma$  by adding the rules  $00 = 0$  and  $\varepsilon 0 = 0$  for any  $\varepsilon \in \Gamma$ . This is in agreement with the invariance of the end 0 with respect to translations in  $\Gamma$ .

The space  $(X, d)$  is a complete, locally compact metric space. For any  $r > 0$  and any  $x \in X$  we denote by  $B(x, r)$  the open ball of center  $x$  and radius  $r$  in the metric space  $X$ .

On the metric space  $(X, d)$  we work with the topology (and uniformity) induced by the distance. For any  $x \in X$  we denote by  $\mathcal{V}(x)$  the topological filter of open neighbourhoods of  $x$ .

#### 3.2 Axioms of dilatation structures

The axioms of a dilatation structure  $(X, d, \delta)$  are listed further. The first axiom is merely a preparation for the next axioms. That is why we counted it as axiom 0.

**A0.** The dilatations

$$\delta_\varepsilon^x : U(x) \rightarrow V_\varepsilon(x)$$

are defined for any  $\varepsilon \in \Gamma, \nu(\varepsilon) \leq 1$ . The sets  $U(x), V_\varepsilon(x)$  are open neighbourhoods of  $x$ . All dilatations are homeomorphisms (invertible, continuous, with continuous inverse).

We suppose that there is a number  $1 < A$  such that for any  $x \in X$  we have

$$\bar{B}_d(x, A) \subset U(x) .$$

We suppose that for all  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \in (0, 1)$ , we have

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset U(x) .$$

There is a number  $B \in (1, A]$  such that for any  $\varepsilon \in \Gamma$  with  $\nu(\varepsilon) \in (1, +\infty)$  the associated dilatation

$$\delta_\varepsilon^x : W_\varepsilon(x) \rightarrow B_d(x, B) ,$$

is injective, invertible on the image. We shall suppose that  $W_\varepsilon(x) \in \mathcal{V}(x)$ , that  $V_{\varepsilon^{-1}}(x) \subset W_\varepsilon(x)$  and that for all  $\varepsilon \in \Gamma_1$  and  $u \in U(x)$  we have

$$\delta_{\varepsilon^{-1}}^x \delta_\varepsilon^x u = u .$$

We have therefore the following string of inclusions, for any  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \leq 1$ , and any  $x \in X$ :

$$B_d(x, \nu(\varepsilon)) \subset \delta_\varepsilon^x B_d(x, A) \subset V_\varepsilon(x) \subset W_{\varepsilon^{-1}}(x) \subset \delta_\varepsilon^x B_d(x, B) .$$

A further technical condition on the sets  $V_\varepsilon(x)$  and  $W_\varepsilon(x)$  will be given just before the axiom A4. (This condition will be counted as part of axiom A0.)

**A1.** We have  $\delta_\varepsilon^x x = x$  for any point  $x$ . We also have  $\delta_1^x = id$  for any  $x \in X$ .

Let us define the topological space

$$\begin{aligned} \text{dom } \delta = \{(\varepsilon, x, y) \in \Gamma \times X \times X : & \quad \text{if } \nu(\varepsilon) \leq 1 \text{ then } y \in U(x) , \\ & \text{else } y \in W_\varepsilon(x)\} \end{aligned}$$

with the topology inherited from the product topology on  $\Gamma \times X \times X$ . Consider also  $Cl(\text{dom } \delta)$ , the closure of  $\text{dom } \delta$  in  $\bar{\Gamma} \times X \times X$  with product topology. The function  $\delta : \text{dom } \delta \rightarrow X$  defined by  $\delta(\varepsilon, x, y) = \delta_\varepsilon^x y$  is continuous. Moreover, it can be continuously extended to  $Cl(\text{dom } \delta)$  and we have

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^x y = x .$$

**A2.** For any  $x \in K$ ,  $\varepsilon, \mu \in \Gamma_1$  and  $u \in \bar{B}_d(x, A)$  we have:

$$\delta_\varepsilon^x \delta_\mu^x u = \delta_{\varepsilon\mu}^x u .$$

**A3.** For any  $x$  there is a function  $(u, v) \mapsto d^x(u, v)$ , defined for any  $u, v$  in the closed ball (in distance  $d$ )  $\bar{B}(x, A)$ , such that

$$\lim_{\varepsilon \rightarrow 0} \sup \left\{ \left| \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x v) - d^x(u, v) \right| : u, v \in \bar{B}_d(x, A) \right\} = 0$$

uniformly with respect to  $x$  in compact set.

**Remark 3.1** *The "distance"  $d^x$  can be degenerated: there might exist  $v, w \in U(x)$  such that  $d^x(v, w) = 0$ .*

For the following axiom to make sense we impose a technical condition on the co-domains  $V_\varepsilon(x)$ : for any compact set  $K \subset X$  there are  $R = R(K) > 0$  and  $\varepsilon_0 = \varepsilon(K) \in (0, 1)$  such that for all  $u, v \in \bar{B}_d(x, R)$  and all  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \in (0, \varepsilon_0)$ , we have

$$\delta_\varepsilon^x v \in W_{\varepsilon^{-1}}(\delta_\varepsilon^x u) .$$

With this assumption the following notation makes sense:

$$\Delta_\varepsilon^x(u, v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x v .$$

The next axiom can now be stated:

**A4.** We have the limit

$$\lim_{\varepsilon \rightarrow 0} \Delta_\varepsilon^x(u, v) = \Delta^x(u, v)$$

uniformly with respect to  $x, u, v$  in compact set.

**Definition 3.2** *A triple  $(X, d, \delta)$  which satisfies A0, A1, A2, A3, but  $d^x$  is degenerate for some  $x \in X$ , is called degenerate dilatation structure.*

*If the triple  $(X, d, \delta)$  satisfies A0, A1, A2, A3 and  $d^x$  is non-degenerate for any  $x \in X$ , then we call it a dilatation structure.*

*If a dilatation structure satisfies A4 then we call it strong dilatation structure.*

### 3.3 Groups with dilatations. Conical groups

Metric tangent spaces sometimes have a group structure which is compatible with dilatations. This structure, of a group with dilatations, is interesting by itself. The notion has been introduced in [2]; we describe it further.

The following description of local uniform groups is slightly non canonical, but is motivated by the case of a Lie group endowed with a Carnot-Caratheodory distance induced by a left invariant distribution (see for example [2]).

We begin with some notations. Let  $G$  be a group. We introduce first the double of  $G$ , as the group  $G^{(2)} = G \times G$  with operation

$$(x, u)(y, v) = (xy, y^{-1}uv) \quad .$$

The operation on the group  $G$ , seen as the function  $op : G^{(2)} \rightarrow G$ ,  $op(x, y) = xy$  is a group morphism. Also the inclusions:

$$i' : G \rightarrow G^{(2)} \quad , \quad i'(x) = (x, e)$$

$$i'' : G \rightarrow G^{(2)} \quad , \quad i''(x) = (x, x^{-1})$$

are group morphisms.

**Definition 3.3** 1.  $G$  is an uniform group if we have two uniformity structures, on  $G$  and  $G \times G$ , such that  $op$ ,  $i'$ ,  $i''$  are uniformly continuous.

2. A local action of a uniform group  $G$  on a uniform pointed space  $(X, x_0)$  is a function  $\phi \in W \in \mathcal{V}(e) \mapsto \hat{\phi} : U_\phi \in \mathcal{V}(x_0) \rightarrow V_\phi \in \mathcal{V}(x_0)$  such that:

(a) the map  $(\phi, x) \mapsto \hat{\phi}(x)$  is uniformly continuous from  $G \times X$  (with product uniformity) to  $X$ ,

(b) for any  $\phi, \psi \in G$  there is  $D \in \mathcal{V}(x_0)$  such that for any  $x \in D$   $\phi\psi^{-1}(x)$  and  $\hat{\phi}(\hat{\psi}^{-1}(x))$  make sense and  $\phi\psi^{-1}(x) = \hat{\phi}(\hat{\psi}^{-1}(x))$ .

3. Finally, a local group is an uniform space  $G$  with an operation defined in a neighbourhood of  $(e, e) \subset G \times G$  which satisfies the uniform group axioms locally.

An uniform group, according to the definition (3.3), is a group  $G$  such that left translations are uniformly continuous functions and the left action of  $G$  on itself is uniformly continuous too.

**Definition 3.4** A group with dilatations  $(G, \delta)$  is a local uniform group  $G$  with a local action of  $\Gamma$  (denoted by  $\delta$ ), on  $G$  such that

H0. the limit  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon x = e$  exists and is uniform with respect to  $x$  in a compact neighbourhood of the identity  $e$ .

H1. the limit

$$\beta(x, y) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} ((\delta_\varepsilon x)(\delta_\varepsilon y))$$

is well defined in a compact neighbourhood of  $e$  and the limit is uniform.

H2. the following relation holds

$$\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{-1} ((\delta_\varepsilon x)^{-1}) = x^{-1}$$

where the limit from the left hand side exists in a neighbourhood of  $e$  and is uniform with respect to  $x$ .

These axioms are in fact a particular version of the axioms for a dilatation structure.

**Definition 3.5** A (local) conical group  $N$  is a (local) group with a (local) action of  $\Gamma$  by morphisms  $\delta_\varepsilon$  such that  $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon x = e$  for any  $x$  in a neighbourhood of the neutral element  $e$ .

A conical group is the infinitesimal version of a group with dilatations ([3] proposition 2).

**Proposition 3.6** Under the hypotheses H0, H1, H2  $(G, \beta, \delta)$  is a conical group, with operation  $\beta$  and dilatations  $\delta$ .

Any group with dilatations has an associated dilatation structure on it. In a group with dilatations  $(G, \delta)$  we define dilatations based in any point  $x \in G$  by

$$\delta_\varepsilon^x u = x \delta_\varepsilon (x^{-1} u). \quad (3.3.1)$$

**Definition 3.7** A normed group with dilatations  $(G, \delta, \|\cdot\|)$  is a group with dilatations  $(G, \delta)$  endowed with a continuous norm function  $\|\cdot\| : G \rightarrow \mathbb{R}$  which satisfies (locally, in a neighbourhood of the neutral element  $e$ ) the properties:

- (a) for any  $x$  we have  $\|x\| \geq 0$ ; if  $\|x\| = 0$  then  $x = e$ ,
- (b) for any  $x, y$  we have  $\|xy\| \leq \|x\| + \|y\|$ ,
- (c) for any  $x$  we have  $\|x^{-1}\| = \|x\|$ ,
- (d) the limit  $\lim_{\varepsilon \rightarrow 0} \frac{1}{\nu(\varepsilon)} \|\delta_\varepsilon x\| = \|x\|^N$  exists, is uniform with respect to  $x$  in compact set,
- (e) if  $\|x\|^N = 0$  then  $x = e$ .

It is easy to see that if  $(G, \delta, \|\cdot\|)$  is a normed group with dilatations then  $(G, \beta, \delta, \|\cdot\|^N)$  is a normed conical group. The norm  $\|\cdot\|^N$  satisfies the stronger form of property (d) definition 3.7: for any  $\varepsilon > 0$

$$\|\delta_\varepsilon x\|^N = \varepsilon \|x\|^N \quad .$$

In a normed group with dilatations we have a natural left invariant distance given by

$$d(x, y) = \|x^{-1}y\| \quad . \quad (3.3.2)$$

The following result is theorem 15 [3].

**Theorem 3.8** *Let  $(G, \delta, \|\cdot\|)$  be a locally compact normed group with dilatations. Then  $(G, \delta, d)$  is a dilatation structure, where  $\delta$  are the dilatations defined by (3.3.1) and the distance  $d$  is induced by the norm as in (3.3.2).*

### 3.4 Carnot groups

Normed conical groups generalize the notion of Carnot groups. A simply connected Lie group whose Lie algebra admits a positive graduation is also called a Carnot group. It is in particular nilpotent. Such objects appear in sub-riemannian geometry as models of tangent spaces, cf. [1], [7], [8].

**Definition 3.9** *A Carnot (or stratified nilpotent) group is a pair  $(N, V_1)$  consisting of a real connected simply connected group  $N$  with a distinguished subspace  $V_1$  of the Lie algebra  $\text{Lie}(N)$ , such that the following direct sum decomposition occurs:*

$$n = \sum_{i=1}^m V_i, \quad V_{i+1} = [V_1, V_i]$$

*The number  $m$  is the step of the group. The number  $Q = \sum_{i=1}^m i \dim V_i$  is called the homogeneous dimension of the group.*

Because the group is nilpotent and simply connected, the exponential mapping is a diffeomorphism. We shall identify the group with the algebra, if is not locally otherwise stated.

The structure that we obtain is a set  $N$  endowed with a Lie bracket and a group multiplication operation, related by the Baker-Campbell-Hausdorff formula. Remark that the group operation is polynomial.

Any Carnot group admits a one-parameter family of dilatations. For any  $\varepsilon > 0$ , the associated dilatation is:

$$x = \sum_{i=1}^m x_i \mapsto \delta_\varepsilon x = \sum_{i=1}^m \varepsilon^i x_i$$

Any such dilatation is a group morphism and a Lie algebra morphism.

In fact the class of Carnot groups is characterised by the existence of dilatations (see Folland-Stein [4], section 1).

**Proposition 3.10** *Suppose that the Lie algebra  $\mathfrak{g}$  admits an one parameter group  $\varepsilon \in (0, +\infty) \mapsto \delta_\varepsilon$  of simultaneously diagonalisable Lie algebra isomorphisms. Then  $\mathfrak{g}$  is the algebra of a Carnot group.*

We shall construct a norm on a Carnot group  $N$ . First pick an euclidean norm  $\|\cdot\|$  on  $V_1$ . We shall endow the group  $N$  with a structure of a sub-Riemannian manifold now. For this take the distribution obtained from left translates of the space  $V_1$ . The metric on that distribution is obtained by left translation of the inner product restricted to  $V_1$ .

Because  $V_1$  generates (the algebra)  $N$  then any element  $x \in N$  can be written as a product of elements from  $V_1$ . An useful lemma is the following (slight reformulation of Lemma 1.40, Folland, Stein [4]).

**Lemma 3.11** *Let  $N$  be a Carnot group and  $X_1, \dots, X_p$  an orthonormal basis for  $V_1$ . Then there is a natural number  $M$  and a function  $g : \{1, \dots, M\} \rightarrow \{1, \dots, p\}$  such that any element  $x \in N$  can be written as:*

$$x = \prod_{i=1}^M \exp(t_i X_{g(i)}) \quad (3.4.3)$$

Moreover, if  $x$  is sufficiently close (in Euclidean norm) to 0 then each  $t_i$  can be chosen such that  $|t_i| \leq C\|x\|^{1/m}$

As a consequence we get:

**Corollary 3.12** *The Carnot-Carathéodory distance*

$$d(x, y) = \inf \left\{ \int_0^1 \|c^{-1}\dot{c}\| dt : c(0) = x, c(1) = y, \right. \\ \left. c^{-1}(t)\dot{c}(t) \in V_1 \text{ for a.e. } t \in [0, 1] \right\}$$

is finite for any two  $x, y \in N$ . The distance is obviously left invariant, thus it induces a norm on  $N$ .

### 3.5 Contractible groups

Conical groups are particular examples of (local) contraction groups.

**Definition 3.13** *A contraction group is a pair  $(G, \alpha)$ , where  $G$  is a topological group with neutral element denoted by  $e$ , and  $\alpha \in \text{Aut}(G)$  is an automorphism of  $G$  such that:*

- $\alpha$  is continuous, with continuous inverse,
- for any  $x \in G$  we have the limit  $\lim_{n \rightarrow \infty} \alpha^n(x) = e$ .

We shall be interested in locally compact contraction groups  $(G, \alpha)$ , such that  $\alpha$  is compactly contractive, that is: for each compact set  $K \subset G$  and open set  $U \subset G$ , with  $e \in U$ , there is  $N(K, U) \in \mathbb{N}$  such that for any  $x \in K$  and  $n \in \mathbb{N}$ ,  $n \geq N(K, U)$ , we have  $\alpha^n(x) \in U$ . If  $G$  is locally compact then a necessary and sufficient condition for  $(G, \alpha)$  to be compactly contractive is:  $\alpha$  is an uniform contraction, that is each identity neighbourhood of  $G$  contains an  $\alpha$ -invariant neighbourhood.

A conical group is an example of a locally compact, compactly contractive, contraction group. Indeed, it suffices to associate to a conical group  $(G, \delta)$  the contraction group  $(G, \delta_\varepsilon)$ , for a fixed  $\varepsilon \in \Gamma$  with  $\nu(\varepsilon) < 1$ .

Conversely, to any contraction group  $(G, \alpha)$ , which is locally compact and compactly contractive, associate the conical group  $(G, \delta)$ , with  $\Gamma = \left\{ \frac{1}{2^n} : n \in \mathbb{N} \right\}$  and for any  $n \in \mathbb{N}$  and  $x \in G$

$$\delta_{\frac{1}{2^n}} x = \alpha^n(x) \quad .$$

Finally, a local conical group has only locally the structure of a contraction group. The structure of contraction groups is known in some detail, due to Siebert [9], Wang [10], Glöckner and Willis [6], Glöckner [5] and references therein.

For this paper the following results are of interest. We begin with the definition of a contracting automorphism group [9], definition 5.1.

**Definition 3.14** *Let  $G$  be a locally compact group. An automorphism group on  $G$  is a family  $T = (\tau_t)_{t>0}$  in  $\text{Aut}(G)$ , such that  $\tau_t \tau_s = \tau_{ts}$  for all  $t, s > 0$ .*

*The contraction group of  $T$  is defined by*

$$C(T) = \left\{ x \in G : \lim_{t \rightarrow 0} \tau_t(x) = e \right\} \quad .$$

*The automorphism group  $T$  is contractive if  $C(T) = G$ .*

It is obvious that a contractive automorphism group  $T$  induces on  $G$  a structure of conical group. Conversely, any conical group with  $\Gamma = (0, +\infty)$  has an associated contractive automorphism group (the group of dilatations based at the neutral element).

Further is proposition 5.4 [9].

**Proposition 3.15** *For a locally compact group  $G$  the following assertions are equivalent:*

- (i)  $G$  admits a contractive automorphism group;
- (ii)  $G$  is a simply connected Lie group whose Lie algebra admits a positive graduation.

## 4 Properties of dilatation structures

### 4.1 First properties

The sum, difference, inverse operations induced by a dilatation structure give to the space  $X$  almost the structure of an affine space. We collect some results from [3] section 4.2 , regarding the properties of these operations.

**Theorem 4.1** *Let  $(X, d, \delta)$  be a dilatation structure. Then, for any  $x \in X$ ,  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) < 1$ , we have:*

- (a) for any  $u \in U(x)$ ,  $\Sigma_\varepsilon^x(x, u) = u$  .
- (b) for any  $u \in U(x)$  the functions  $\Sigma_\varepsilon^x(u, \cdot)$  and  $\Delta_\varepsilon^x(u, \cdot)$  are inverse one to another.
- (c) the inverse function is shifted involutive: for any  $u \in U(x)$ ,

$$\text{inv}_\varepsilon^{\delta_\varepsilon^x u} \text{inv}_\varepsilon^x(u) = u \quad .$$

- (d) the sum operation is shifted associative: for any  $u, v, w$  sufficiently close to  $x$  we have

$$\Sigma_\varepsilon^x(u, \Sigma_\varepsilon^{\delta_\varepsilon^x u}(v, w)) = \Sigma_\varepsilon^x(\Sigma_\varepsilon^x(u, v), w) \quad .$$

- (e) the difference, inverse and sum operations are related by

$$\Delta_\varepsilon^x(u, v) = \Sigma_\varepsilon^{\delta_\varepsilon^x u}(\text{inv}_\varepsilon^x(u), v) \quad ,$$

for any  $u, v$  sufficiently close to  $x$ .

(f) for any  $u, v$  sufficiently close to  $x$  and  $\mu \in \Gamma$ ,  $\nu(\mu) < 1$ , we have:

$$\Delta_\varepsilon^x(\delta_\mu^x u, \delta_\mu^x v) = \delta_\mu^{\delta_\varepsilon^x u} \Delta_{\varepsilon\mu}^x(u, v) \quad .$$

## 4.2 Tangent bundle

A reformulation of parts of theorems 6,7 [3] is the following.

**Theorem 4.2** *A dilatation structure  $(X, d, \delta)$  has the following properties.*

(a) *For all  $x \in X$ ,  $u, v \in X$  such that  $d(x, u) \leq 1$  and  $d(x, v) \leq 1$  and all  $\mu \in (0, A)$  we have:*

$$d^x(u, v) = \frac{1}{\mu} d^x(\delta_\mu^x u, \delta_\mu^x v) \quad .$$

*We shall say that  $d^x$  has the cone property with respect to dilatations.*

(b) *The metric space  $(X, d)$  admits a metric tangent space at  $x$ , for any point  $x \in X$ . More precisely we have the following limit:*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \sup \{ |d(u, v) - d^x(u, v)| : d(x, u) \leq \varepsilon, d(x, v) \leq \varepsilon \} = 0 \quad .$$

For the next theorem (composite of results in theorems 8, 10 [3]) we need the previously introduced notion of a normed conical (local) group.

**Theorem 4.3** *Let  $(X, d, \delta)$  be a strong dilatation structure. Then for any  $x \in X$  the triple  $(U(x), \Sigma^x, \delta^x)$  is a normed local conical group, with the norm induced by the distance  $d^x$ .*

The conical group  $(U(x), \Sigma^x, \delta^x)$  can be regarded as the tangent space of  $(X, d, \delta)$  at  $x$ . Further will be denoted by:  $T_x X = (U(x), \Sigma^x, \delta^x)$ .

**Definition 4.4** *Let  $(X, \delta, d)$  be a dilatation structure and  $x \in X$  a point. In a neighbourhood  $U(x)$  of  $x$ , for any  $\mu \in (0, 1)$  we defined the distances:*

$$(\delta^x, \mu)(u, v) = \frac{1}{\mu} d(\delta_\mu^x u, \delta_\mu^x v).$$

**Proposition 4.5** *Let  $(X, \delta, d)$  be a (strong) dilatation structure. For any  $u, v \in U(x)$  let us define*

$$\hat{\delta}_\varepsilon^u v = \Sigma_\mu^x(u, \delta_\varepsilon^{\delta_\mu^x u} \Delta_\mu^x(u, v)) = \delta_{\mu^{-1}}^x \delta_\varepsilon^{\delta_\mu^x u} \delta_\mu^x v.$$

*Then  $(U(x), \hat{\delta}, (\delta^x, \mu))$  is a (strong) dilatation structure.*

**Proof.** We have to check the axioms. The first part of axiom A0 is an easy consequence of theorem 4.2 for  $(X, \delta, d)$ . The second part of A0, A1 and A2 are true based on simple computations.

The first interesting fact is related to axiom A3. Let us compute, for  $v, w \in U(x)$ ,

$$\begin{aligned} \frac{1}{\varepsilon}(\delta^x, \mu)(\hat{\delta}_\varepsilon^u v, \hat{\delta}_\varepsilon^u w) &= \frac{1}{\varepsilon\mu}d(\delta_\mu^x \hat{\delta}_\varepsilon^u v, \delta_\mu^x \hat{\delta}_\varepsilon^u w) = \\ &= \frac{1}{\varepsilon\mu}d(\delta_\varepsilon^{\delta_\mu^x u} \delta_\mu^x v, \delta_\varepsilon^{\delta_\mu^x u} \delta_\mu^x w) = \frac{1}{\varepsilon\mu}d(\delta_\varepsilon^{\delta_\mu^x u} \Delta_\mu^x(u, v), \delta_\varepsilon^{\delta_\mu^x u} \Delta_\mu^x(u, w)) = \\ &= (\delta_\varepsilon^{\delta_\mu^x u}, \varepsilon\mu)(\Delta_\mu^x(u, v), \Delta_\mu^x(u, w)). \end{aligned}$$

The axiom A3 is then a consequence of axiom A3 for  $(X, \delta, d)$  and we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon}(\delta^x, \mu)(\hat{\delta}_\varepsilon^u v, \hat{\delta}_\varepsilon^u w) = d^{\delta_\mu^x u}(\Delta_\mu^x(u, v), \Delta_\mu^x(u, w)).$$

The axiom A4 is also a straightforward consequence of A4 for  $(X, \delta, d)$  and is left to the reader.  $\square$

The proof of the following proposition is an easy computation, of the same type as in the lines above, therefore we shall not write it here.

**Proposition 4.6** *With the same notations as in proposition 4.5, the transformation  $\Sigma_\mu^x(u, \cdot)$  is an isometry from  $(\delta_\mu^{\delta_\mu^x u}, \mu)$  to  $(\delta^x, \mu)$ . Moreover, we have  $\Sigma_\mu^x(u, \delta_\mu^x u) = u$ .*

These two propositions show that on a dilatation structure we almost have translations (the operators  $\Sigma_\varepsilon^x(u, \cdot)$ ), which are almost isometries (that is, not with respect to the distance  $d$ , but with respect to distances of type  $(\delta^x, \mu)$ ). It is almost as if we were working with a normed conical group, only that we have to use families of distances and to make small shifts in the tangent space (as in the last formula in the proof of proposition 4.5).

### 4.3 Topological considerations

In this subsection we compare various topologies and uniformities related to a dilatation structure.

The axiom A3 implies that for any  $x \in X$  the function  $d^x$  is continuous, therefore open sets with respect to  $d^x$  are open with respect to  $d$ .

If  $(X, d)$  is separable and  $d^x$  is non degenerate then  $(U(x), d^x)$  is also separable and the topologies of  $d$  and  $d^x$  are the same. Therefore  $(U(x), d^x)$  is also locally compact (and a set is compact with respect to  $d^x$  if and only if it is compact with respect to  $d$ ).

If  $(X, d)$  is separable and  $d^x$  is non degenerate then the uniformities induced by  $d$  and  $d^x$  are the same. Indeed, let  $\{u_n : n \in \mathbb{N}\}$  be a dense set in  $U(x)$ , with  $x_0 = x$ . We can embed  $(U(x), (\delta^x, \varepsilon))$  isometrically in the separable Banach space  $l^\infty$ , for any  $\varepsilon \in (0, 1)$ , by the function

$$\phi_\varepsilon(u) = \left( \frac{1}{\varepsilon} d(\delta_\varepsilon^x u, \delta_\varepsilon^x x_n) - \frac{1}{\varepsilon} d(\delta_\varepsilon^x x, \delta_\varepsilon^x x_n) \right)_n .$$

A reformulation of point (a) in theorem 4.2 is that on compact sets  $\phi_\varepsilon$  uniformly converges to the isometric embedding of  $(U(x), d^x)$

$$\phi(u) = (d^x(u, x_n) - d^x(x, x_n))_n .$$

Remark that the uniformity induced by  $(\delta, \varepsilon)$  is the same as the uniformity induced by  $d$ , and that it is the same induced from the uniformity on  $l^\infty$  by the embedding  $\phi_\varepsilon$ . We proved that the uniformities induced by  $d$  and  $d^x$  are the same.

From previous considerations we deduce the following characterisation of tangent spaces asociated to a dilatation structure.

**Corollary 4.7** *Let  $(X, d, \delta)$  be a strong dilatation structure with group  $\Gamma = (0, +\infty)$ . Then for any  $x \in X$  the local group  $(U(x), \Sigma^x)$  is locally a simply connected Lie group whose Lie algebra admits a positive graduation (a Carnot group).*

**Proof.** Use the facts:  $(U(x), \Sigma^x)$  is a locally compact group (from previous topological considerations) which admits  $\delta^x$  as a contractive automorphism group (from theorem 4.3). Apply then Siebert proposition 3.15.  $\square$

## 5 Linearity and dilatation structures

**Definition 5.1** *Let  $(X, d, \delta)$  be a dilatation structure. A transformation  $A : X \rightarrow X$  is linear if it is Lipschitz and it commutes with dilatations in the following sense: for any  $x \in X$ ,  $u \in U(x)$  and  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) < 1$ , if  $A(u) \in U(A(x))$  then*

$$A\delta_\varepsilon^x = \delta^{A(x)} A(u) \quad .$$

*The group of linear transformations, denoted by  $GL(X, d, \delta)$  is formed by all invertible and bi-lipschitz linear transformations of  $X$ .*

$GL(X, d, \delta)$  is indeed a (local) group. In order to see this we start from the remark that if  $A$  is Lipschitz then there exists  $C > 0$  such that for all  $x \in X$

and  $u \in B(x, C)$  we have  $A(u) \in U(A(x))$ . The inverse of  $A \in GL(X, d, \delta)$  is then linear. Same considerations apply for the composition of two linear, bi-lipschitz and invertible transformations.

In the particular case of the first subsection of this paper, namely  $X$  finite dimensional real, normed vector space,  $d$  the distance given by the norm,  $\Gamma = (0, +\infty)$  and dilatations  $\delta_\varepsilon^x u = x + \varepsilon(u - x)$ , a linear transformations in the sense of definition 5.1 is an affine transformation of the vector space  $X$ .

**Proposition 5.2** *Let  $(X, d, \delta)$  be a dilatation structure and  $A : X \rightarrow X$  a linear transformation. Then:*

(a) *for all  $x \in X$ ,  $u, v \in U(x)$  sufficiently close to  $x$ , we have:*

$$A \Sigma_\varepsilon^x(u, v) = \Sigma_\varepsilon^{A(x)}(A(u), A(v)) \quad .$$

(b) *for all  $x \in X$ ,  $u \in U(x)$  sufficiently close to  $x$ , we have:*

$$A \operatorname{inv}^x(u) = \operatorname{inv}^{A(x)} A(u) \quad .$$

**Proof.** Straightforward, just use the commutation with dilatations.  $\square$

## 5.1 Differentiability of linear transformations

In this subsection we briefly recall the notion of differentiability associated to dilatation structures (section 7.2 [3]). Then we apply it for linear transformations.

First we need the natural definition below.

**Definition 5.3** *Let  $(N, \delta)$  and  $(M, \bar{\delta})$  be two conical groups. A function  $f : N \rightarrow M$  is a conical group morphism if  $f$  is a group morphism and for any  $\varepsilon > 0$  and  $u \in N$  we have  $f(\delta_\varepsilon u) = \bar{\delta}_\varepsilon f(u)$ .*

The definition of derivative with respect to dilatations structures follows.

**Definition 5.4** *Let  $(X, \delta, d)$  and  $(Y, \bar{\delta}, \bar{d})$  be two strong dilatation structures and  $f : X \rightarrow Y$  be a continuous function. The function  $f$  is differentiable in  $x$  if there exists a conical group morphism  $Q^x : T_x X \rightarrow T_{f(x)} Y$ , defined on a neighbourhood of  $x$  with values in a neighbourhood of  $f(x)$  such that*

$$\lim_{\varepsilon \rightarrow 0} \sup \left\{ \frac{1}{\varepsilon} \bar{d} \left( f(\delta_\varepsilon^x u), \bar{\delta}_\varepsilon^{f(x)} Q^x(u) \right) : d(x, u) \leq \varepsilon \right\} = 0, \quad (5.1.1)$$

The morphism  $Q^x$  is called the derivative of  $f$  at  $x$  and will be sometimes denoted by  $Df(x)$ .

The function  $f$  is uniformly differentiable if it is differentiable everywhere and the limit in (5.1.1) is uniform in  $x$  in compact sets.

The following proposition has then a straightforward proof.

**Proposition 5.5** *Let  $(X, d, \delta)$  be a strong dilatation structure and  $A : X \rightarrow X$  a linear transformation. Then  $A$  is uniformly differentiable and the derivative equals  $A$ .*

## 5.2 Linearity of strong dilatation structures

Remark that for general dilatation structures the "translations"  $\Sigma_\varepsilon^x(u, \cdot)$  are not linear. Nevertheless, they commute with dilatation in a known way, according to point (f) theorem 4.1. This is important, because the transformations  $\Sigma_\varepsilon^x(u, \cdot)$  really behave as translations, as explained in subsection 4.1.

The reason for which translations are not linear is that dilatations are generally not linear. Before giving the next definition we need to establish a simple estimate. Let  $K \subset X$  be compact, non empty set. Then there is a constant  $C(K) > 0$ , depending on the set  $K$  such that for any  $\varepsilon, \mu \in \Gamma$  with  $\nu(\varepsilon), \nu(\mu) \in (0, 1]$  and any  $x, y, z \in K$  with  $d(x, y), d(x, z), d(y, z) \leq C(K)$  we have

$$\delta_\mu^y z \in V_\varepsilon(x) \quad , \quad \delta_\varepsilon^x z \in V_\mu(\delta_\varepsilon^x y) \quad .$$

Indeed, this is coming from the uniform (with respect to  $K$ ) estimates:

$$d(\delta_\varepsilon^x y, \delta_\varepsilon^x z) \leq \varepsilon d^x(y, z) + \varepsilon \mathcal{O}(\varepsilon) \quad ,$$

$$d(x, \delta_\mu^y z) \leq d(x, y) + d(y, \delta_\mu^y z) \leq d(x, y) + \mu d^y(y, z) + \mu \mathcal{O}(\mu) \quad .$$

**Definition 5.6** *A property  $\mathcal{P}(x_1, x_2, x_3, \dots)$  holds for  $x_1, x_2, x_3, \dots$  sufficiently closed if for any compact, non empty set  $K \subset X$ , there is a positive constant  $C(K) > 0$  such that  $\mathcal{P}(x_1, x_2, x_3, \dots)$  is true for any  $x_1, x_2, x_3, \dots \in K$  with  $d(x_i, x_j) \leq C(K)$ .*

For example, we may say that the expressions

$$\delta_\varepsilon^x \delta_\mu^y z \quad , \quad \delta_\mu^{\delta_\varepsilon^x y} \delta_\varepsilon^x z$$

are well defined for any  $x, y, z \in X$  sufficiently closed and for any  $\varepsilon, \mu \in \Gamma$  with  $\nu(\varepsilon), \nu(\mu) \in (0, 1]$ .

**Definition 5.7** A dilatation structure  $(X, d, \delta)$  is linear if for any  $\varepsilon, \mu \in \Gamma$  such that  $\nu(\varepsilon), \nu(\mu) \in (0, 1]$ , and for any  $x, y, z \in X$  sufficiently closed we have

$$\delta_\varepsilon^x \delta_\mu^y z = \delta_\mu^{\delta_\varepsilon^x y} \delta_\varepsilon^x z \quad .$$

Linear dilatation structures are very particular dilatation structures. The next proposition gives a family of examples of linear dilatation structures.

**Proposition 5.8** The dilatation structure associated to a normed conical group is linear.

**Proof.** Indeed, for the dilatation structure associated to a normed conical group we have, with the notations from definition 5.7:

$$\begin{aligned} \delta_\mu^{\delta_\varepsilon^x y} \delta_\varepsilon^x z &= (x \delta_\varepsilon(x^{-1}y)) \delta_\mu(\delta_\varepsilon(y^{-1}x) x^{-1} x \delta_\varepsilon(x^{-1}z)) = \\ &= (x \delta_\varepsilon(x^{-1}y)) \delta_\mu(\delta_\varepsilon(y^{-1}x) \delta_\varepsilon(x^{-1}z)) = (x \delta_\varepsilon(x^{-1}y)) \delta_\mu(\delta_\varepsilon(y^{-1}z)) = \\ &= x (\delta_\varepsilon(x^{-1}y) \delta_\varepsilon \delta_\mu(y^{-1}z)) = x \delta_\varepsilon(x^{-1}y \delta_\mu(y^{-1}z)) = \delta_\varepsilon^x \delta_\mu^y z \quad . \end{aligned}$$

Therefore the dilatation structure is linear.  $\square$

In the proposition below we give a relation, true for linear dilatation structures, with an interesting interpretation. Let us think in affine terms: for closed points  $x, u, v$ , we think about let us denote  $w = \Sigma_\varepsilon^x(u, v)$ . We may think that the "vector"  $(x, w)$  is (approximatively, due to the parameter  $\varepsilon$ ) the sum of the vectors  $(x, u)$  and  $(x, v)$ , based at  $x$ . Denote also  $w' = \Delta_\varepsilon^u(x, v)$ ; then the "vector"  $(u, w')$  is (approximatively) equal to the difference between the vectors  $(u, v)$  and  $(u, x)$ , based at  $u$ . In a classical affine space we would have  $w = w'$ . The same is true for a linear dilatation structure.

**Proposition 5.9** For a linear dilatation structure  $(X, \delta, d)$ , for any  $x, u, v \in X$  sufficiently closed and for any  $\varepsilon \in \Gamma$ ,  $\nu(\varepsilon) \leq 1$ , we have:

$$\Sigma_\varepsilon^x(u, v) = \Delta_\varepsilon^u(x, v) \quad .$$

**Proof.** We have the following string of equalities, by using twice the linearity of the dilatation structure:

$$\begin{aligned} \Sigma_\varepsilon^x(u, v) &= \delta_{\varepsilon^{-1}}^x \delta_\varepsilon^{\delta_\varepsilon^x u} v = \delta_\varepsilon^u \delta_{\varepsilon^{-1}}^x v = \\ &= \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^u x} \delta_\varepsilon^u v = \Delta_\varepsilon^u(x, v) \quad . \end{aligned}$$

The proof is done.  $\square$

The following expression:

$$Lin(x, y, z; \varepsilon, \mu) = d(\delta_\varepsilon^x \delta_\mu^y z, \delta_\mu^{\delta_\varepsilon^x y} \delta_\varepsilon^x z) \quad (5.2.2)$$

is a measure of lack of linearity, for a general dilatation structure. The next theorem shows that, infinitesimally, any dilatation structure is linear.

**Theorem 5.10** *Let  $(X, d, \delta)$  be a strong dilatation structure. Then for any  $x, y, z \in X$  sufficiently close we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} Lin(x, \delta_\varepsilon^x y, \delta_\varepsilon^x z; \varepsilon, \varepsilon) = 0 \quad . \quad (5.2.3)$$

**Proof.** From the hypothesis of the theorem we have:

$$\begin{aligned} \frac{1}{\varepsilon^2} Lin(x, \delta_\varepsilon^x y, \delta_\varepsilon^x z; \varepsilon, \varepsilon) &= \frac{1}{\varepsilon^2} d(\delta_\varepsilon^x \delta_\varepsilon^{\delta_\varepsilon^x y} z, \delta_\varepsilon^{\delta_\varepsilon^x \delta_\varepsilon^x y} \delta_\varepsilon^x z) = \\ &= \frac{1}{\varepsilon^2} d(\delta_{\varepsilon^2}^x \Sigma_\varepsilon^x(y, z), \delta_{\varepsilon^2}^x \delta_{\varepsilon^{-2}}^x \delta_\varepsilon^{\delta_\varepsilon^x y} \delta_\varepsilon^x z) = \\ &= \frac{1}{\varepsilon^2} d(\delta_{\varepsilon^2}^x \Sigma_\varepsilon^x(y, z), \delta_{\varepsilon^2}^x \Sigma_{\varepsilon^2}^x(y, \Delta_\varepsilon^x(\delta_\varepsilon^x y, z))) = \\ &= \mathcal{O}(\varepsilon^2) + d^x(\Sigma_\varepsilon^x(y, z), \Sigma_{\varepsilon^2}^x(y, \Delta_\varepsilon^x(\delta_\varepsilon^x y, z))) \quad . \end{aligned}$$

The dilatation structure satisfies A4, therefore as  $\varepsilon$  goes to 0 we obtain:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} Lin(x, \delta_\varepsilon^x y, \delta_\varepsilon^x z; \varepsilon, \varepsilon) &= d^x(\Sigma^x(y, z), \Sigma^x(y, \Delta^x(x, z))) = \\ &= d^x(\Sigma^x(y, z), \Sigma^x(y, z)) = 0 \quad . \quad \square \end{aligned}$$

The linearity of translations  $\Sigma_\varepsilon^x$  is related to the linearity of the dilatation structure, as described in the theorem below, point (a). As a consequence, we prove at point (b) that a linear and strong dilatation structure comes from a conical group.

**Theorem 5.11** *Let  $(X, d, \delta)$  be a dilatation structure.*

- (a) *If the dilatation structure is linear then all transformations  $\Delta_\varepsilon^x(u, \cdot)$  are linear for any  $u \in X$ .*
- (b) *If the dilatation structure is strong (satisfies A4) then it is linear if and only if the dilatations come from the dilatation structure of a conical group, precisely for any  $x \in X$  there is an open neighbourhood  $D \subset X$  of  $x$  such that  $(\overline{D}, d^x, \delta)$  is the same dilatation structure as the dilatation structure of the tangent space of  $(X, d, \delta)$  at  $x$ .*

**Proof.** (a) If dilatations are linear, then let  $\varepsilon, \mu \in \Gamma$ ,  $\nu(\varepsilon), \nu(\mu) \leq 1$ , and  $x, y, u, v \in X$  such that the following computations make sense. We have:

$$\Delta_\varepsilon^x(u, \delta_\mu^y v) = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x \delta_\mu^y v \quad .$$

Let  $A_\varepsilon = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u}$ . We compute:

$$\delta_\mu^{\Delta_\varepsilon^x(u, y)} \Delta_\varepsilon^x(u, v) = \delta_\mu^{A_\varepsilon \delta_\varepsilon^x y} A_\varepsilon \delta_\varepsilon^x v \quad .$$

We use twice the linearity of dilatations:

$$\delta_\mu^{\Delta_\varepsilon^x(u, y)} \Delta_\varepsilon^x(u, v) = A_\varepsilon \delta_\mu^{\delta_\varepsilon^x y} \delta_\varepsilon^x v = \delta_{\varepsilon^{-1}}^{\delta_\varepsilon^x u} \delta_\varepsilon^x \delta_\mu^y v \quad .$$

We proved that:

$$\Delta_\varepsilon^x(u, \delta_\mu^y v) = \delta_\mu^{\Delta_\varepsilon^x(u, y)} \Delta_\varepsilon^x(u, v) \quad ,$$

which is the conclusion of the part (a).

(b) Suppose that the dilatation structure is strong. If dilatations are linear, then by point (a) the transformations  $\Delta_\varepsilon^x(u, \cdot) \delta$  are linear for any  $u \in X$ . Then, with notations made before, for  $y = u$  we get

$$\Delta_\varepsilon^x(u, \delta_\mu^u v) = \delta_\mu^{\delta_\varepsilon^x u} \Delta_\varepsilon^x(u, v) \quad ,$$

which implies

$$\delta_\mu^u v = \Sigma_\varepsilon^x(u, \delta_\mu^x \Delta_\varepsilon^x(u, v)) \quad .$$

We pass to the limit with  $\varepsilon \rightarrow 0$  and we obtain:

$$\delta_\mu^u v = \Sigma^x(u, \delta_\mu^x \Delta^x(u, v)) \quad .$$

We recognize at the right hand side the dilatations associated to the conical group  $T_x X$ .

By proposition 5.8 the opposite implication is straightforward, because the dilatation structure of any conical group is linear.  $\square$

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